

# Inapproximability of Power Allocation with Inelastic Demands in AC Electric Systems and Networks

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**Abstract**—A challenge in future smart grid is how to efficiently allocate power among customers considering inelastic demands, when the power supply is constrained by the network or generation capacities. This problem is an extension to the classical knapsack problem in a way that the item values are expressed as non-positive real or complex numbers representing power demands, rather than positive real numbers. The objective is to maximize the total utility of the customers. Recently in Chau-Elbassioni-Khonji [AAMAS 14], a PTAS was presented for the case where the maximum phase angle between any pair of power demands is  $\phi \leq \frac{\pi}{2}$ ; and a bi-criteria FPTAS when  $\frac{\pi}{2} < \phi \leq \pi - \varepsilon$ , for any polynomially small  $\varepsilon$ . For  $0 \leq \phi \leq \frac{\pi}{2}$ , Yu and Chau [AAMAS 13] showed that unless P=NP, there is no FPTAS. In this paper, we present important hardness results that close the approximation gap. We show that unless P=NP, there is no  $\alpha$ -approximation for  $\frac{\pi}{2} < \pi \leq \pi - \varepsilon$ , where  $\alpha$  is any number with polynomial length. Moreover, for the case when  $\phi$  is arbitrarily close to  $\pi$ , neither a PTAS nor any bi-criteria approximation algorithm with polynomial guarantees can exist. In this paper, we also present a natural generalization to a networked setting such that each edge in the transmission network can have a capacity constraint. We show that there is no bi-criteria approximation algorithm with polynomial guarantees for this networked setting, even all power demands are real (non-complex) numbers.

**Keywords**— Algorithms, Smart Grid, Complex-Demand Knapsack Problem, Hardness Results.

## I. INTRODUCTION

Existing electricity grid faces massive challenges to deliver the growing demand for power, as well as provide a sustainable and robust supply for electricity. Traditionally, the flow in electricity grid is unidirectional from power plants to consumers. Smart grid is the next generation grid in a form of electricity network utilizing communication technologies to deliver electricity from suppliers to consumers using two-way digital communication that control appliances with precise knowledge about the actual demands. The smart grid should be capable of managing several distributed power generations that utilize renewable energy sources effectively to satisfy consumer demands. These complex challenges are driving the developments of new computational models and efficient algorithms that can optimize the power allocation, coordination and pricing mechanisms seamlessly among different entities.

Although resource allocation mechanisms have been well-studied in many systems from transportation to communication networks, the rise of the smart grid presents a new range

of algorithmic problems, which are a departure from these systems. One of the main differences is the presence of periodic time-varying entities (e.g., current, power, voltage) in AC electric systems, which are often expressed in term of non-positive real, or even complex numbers. In power terminology [6], the real component of the complex number is called the *active* power, the imaginary is known as *reactive* power, and the magnitude as *apparent* power. For example, purely resistive appliances have positive active power and zero reactive power. Appliances and instruments with capacitive or inductive components have non-zero reactive power, depending on the phase lag with the input power. Machinery, such as in factories, has large inductors, and hence has positive power demand. On the contrary, shunt-capacitor equipped electric vehicle charging stations can generate reactive power. A well-known problem in power systems engineering is called *reactive power compensation* [4], which uses locally installed capacitor equipped systems to compensate the demand of reactive power from inductor equipped systems.

Recently, Yu and Chau [11] introduced the *complex-demand knapsack* problem (CKP) to model apparent power allocation considering inelastic demands. In the classical knapsack problem, the objective is to select a set of items, each associated with a value and a size, that maximize the total value subject to the total size constraint. On the other hand, CKP introduces quadratic constraints that capture the magnitude of complex number (i.e., apparent power) subject to the power generation constraint. [11] also showed that unless P=NP, there is no FPTAS when the maximum angle  $\phi$  (or phase shift) between any two demands is  $0 \leq \phi \leq \frac{\pi}{2}$ . More recently, [2] presented a PTAS for this case, and a bi-criteria FPTAS for  $\frac{\pi}{2} < \phi \leq \pi - \varepsilon$ , where  $\varepsilon$  is any polynomially small number. In this paper, we consider the case of  $\phi > \frac{\pi}{2}$  and show that the current FPTAS is the best possible. More precisely, unless P=NP, no approximation algorithm exists without violating the capacity constraint. Additionally, if  $\phi$  is arbitrarily close to  $\pi$ , no bi-criteria approximation exists.

Since power systems are connected by electricity grid, one would wonder how the results can be generalized to a networked setting. In general, the power allocation considering only elastic demands in AC electricity grid is characterized by the optimal power flow (OPF) problem, which is non-convex and has no known efficient general algorithm, despite recent progresses of efficient algorithms for certain topologies [9], [10]. However, the results concerning combinatorial power

allocation with inelastic demands are rather limited thus far. We finally present a hardness result showing that there exists no bi-criteria approximation algorithms for a general class of combinatorial power allocation, even all power demands are real numbers such as in DC (direct-current) electric systems.

The paper is structured as followed. In Sec. II, we briefly present the related works. In Sec. III, we provide the problem definitions and notations needed. Then we present our main hardness results in Sec. IV. We extend our hardness proof to the networked setting in Sec. V, followed by the conclusion in Sec. VI.

## II. RELATED WORK

The knapsack problem has many variants with respect to divisibility of items, copies of items, dimensions of constraints, etc [3], [5], [7]. For *one-dimensional knapsack problem* (1DKP), there is a pseudo-polynomial time algorithm using dynamic programming achieving exact optimization when all item values are integers. There is a simple FPTAS based on rounding and scaling the item utilities, then applying dynamic programming [7].

As to *m-dimensional knapsack problem* (mDKP) with  $m \geq 2$ , a PTAS is given in [5] based on the integer programming formulation. On the other hand, 2DKP is already inapproximable by FPTAS unless  $P = NP$ , by a reduction from EQUIPARTITION [7]. In fact, there is no *efficient polynomial-time approximation scheme* (EPTAS) for 2DKP unless  $W[1] = FPT$  (See [8]). Complex demand knapsack was first introduced by Yu and Chau [11] who obtained a  $\frac{1}{2}$ -approximation for the case where  $0 \leq \phi \leq \frac{\pi}{2}$ . They also proved that no FPTAS exist, by reduction from EQUIPARTITION. Recently, [2] gave a PTAS closing the approximation gap, and a bi-criteria FPTAS for  $\frac{\pi}{2} < \phi < \pi - \varepsilon$ . We also note that [2] extended their results to game theoretic settings. Both algorithms are truthful (a.k.a., incentive compatible), which means that the best strategy for users is to reveal their true parameters (namely, demands and utilities) to the algorithm.

## III. PROBLEM DEFINITIONS AND NOTATIONS

### A. 1-Dimensional Knapsack Problem

The traditional *knapsack problem* considering one capacity constraint (1DKP) is defined as follows.

$$(1DKP) \quad \max_{x_k \in \{0,1\}} \sum_{k \in \mathcal{N}} u_k x_k \quad (1)$$

$$\text{subject to} \quad \sum_{k \in \mathcal{N}} x_k d_k^R \leq C. \quad (2)$$

where  $\mathcal{N}$  is a set of  $n$  users,  $u_k \in \mathbb{R}_+$  is the utility of  $k$ -th user if her real-valued demand (i.e.,  $d_k^R \in \mathbb{R}_+$ ) is satisfied,  $C \in \mathbb{R}_+$  is the capacity of the total satisfiable demand, and  $x_k \in \{0,1\}$  is a decision variable, such that  $x_k = 1$  if  $k$ -th user's demand is satisfied, otherwise  $x_k = 0$ . Note that 1DKP is NP-complete, but has an FPTAS [5], [7].

### B. 2-Dimensional Knapsack Problem

In the setting of AC (alternating current) electric systems, a demand for power can be expressed by a complex value  $d_k = d_k^R + id_k^I \in \mathbb{C}$ . One can impose two separate constraints on the capacity of total satisfiable demand of active power (i.e., the real part) and reactive power (i.e., the imaginary part). This problem can be formulated as a *2-dimensional knapsack problem* (2DKP).

$$(2DKP) \quad \max_{x_k \in \{0,1\}} \sum_{k \in \mathcal{N}} u_k x_k \quad (3)$$

$$\text{subject to} \quad \sum_{k \in \mathcal{N}} x_k d_k^R \leq C^R \quad \text{and} \quad \sum_{k \in \mathcal{N}} x_k d_k^I \leq C^I. \quad (4)$$

2DKP can be generalized to *m-dimensional knapsack problem* (mDKP). mDKP is NP-complete, since 1DKP is a special case. Note that mDKP only has a PTAS, and there is no FPTAS for mDKP, when  $m \geq 2$  (unless  $P = NP$ ) [7].

### C. Complex-demand Knapsack Problem

Our study concerns power allocation under a capacity constraint on the apparent power (i.e., the magnitude of the total satisfiable demand). Throughout this paper, we sometimes denote  $\nu^R \triangleq \text{Re}(\nu)$  as the real part and  $\nu^I \triangleq \text{Im}(\nu)$  as the imaginary part of a given complex number  $\nu$ .  $|\nu|$  denotes the magnitude of  $\nu$ .

Complex-demand knapsack problem (CKP) is defined as follows:

$$(CKP) \quad \max_{x_k \in \{0,1\}} \sum_{k \in \mathcal{N}} u_k x_k \quad (5)$$

$$\text{subject to} \quad \left| \sum_{k \in \mathcal{N}} d_k x_k \right| \leq C. \quad (6)$$

where  $d_k = d_k^R + id_k^I \in \mathbb{C}$  is the *complex-valued* demand of power for  $k$ -th user,  $C \in \mathbb{R}_+$  is a real-valued capacity of total satisfiable demand in apparent power. Fig. 1 illustrates a high-level view of the CKP problem. The power plant has limited capacity  $C$  that can not be exceeded by any set of supplied demands, and the objective is to maximize the total utility.  $u_k$  can be interpreted as the amount of money user  $k \in \mathcal{N}$  is willing to pay if her demand  $d_k$  is served.

Fig. 2 gives a pictorial representation of a feasible solution to CKP. The half circle represents Constraint (6) for which the magnitude of any supplied demands cannot exceed it.

Evidently, CKP is also NP-complete, because 1DKP is a special case when we set all  $d_k^I = 0$ . We note that the problem is invariant, when the arguments of all demands are rotated by the same angle. Without loss of generality, we assume that one of the demands, say  $d_1$ , is aligned along the positive real axis, and define a class of sub-problems for CKP, by restricting the maximum phase angle (i.e., the argument) that any other demand makes with  $d_1$ . In particular, we will write  $CKP[\phi_1, \phi_2]$  for the restriction of problem CKP subject to  $\phi_1 \leq \max_{k \in \mathcal{N}} \arg(d_k) \leq \phi_2$ , where  $\arg(d_k) \geq 0$  for all  $k \in \mathcal{N}$  (see Fig. 3 for an illustration). We remark that in realistic setting of power systems, the active power demand

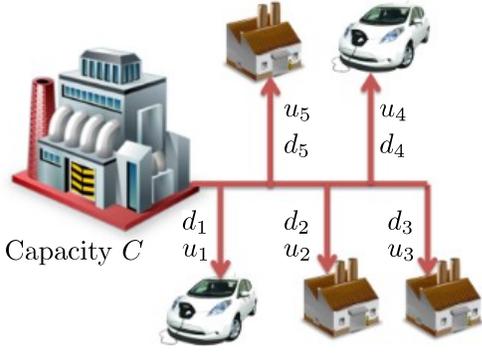


Fig. 1: Each user declares her power demand and utility to the coordinator.

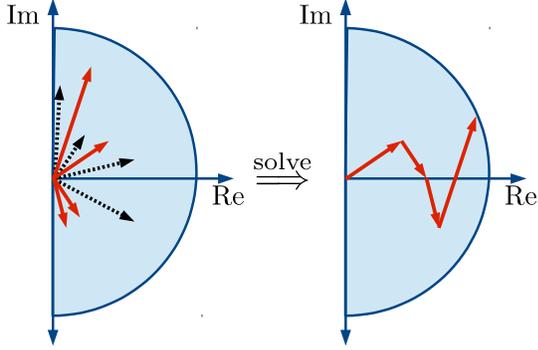


Fig. 2: The left figure shows the set of all demands  $\{d_k\}$ . Red vectors represent a feasible solution to CKP such that the total magnitude of the red demands remain within radius  $C$  as shown in the right figure.

is positive (i.e.,  $d_k^R \geq 0$ ), but the power factor (i.e.,  $\frac{d_k^R}{|d_k|}$ ) is bounded by a certain threshold [1], which is equivalent to restricting the argument of complex-valued demands.

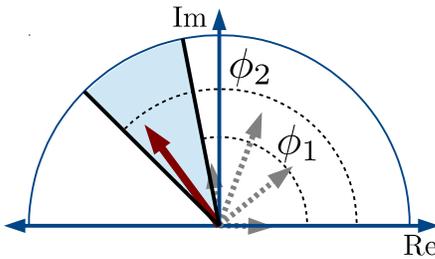


Fig. 3: The shaded area correspond to a class of instances where the maximum  $\arg_k(d_k)$  resides, which is denoted by  $\text{CKP}[\phi_1, \phi_2]$ .

#### D. Networked Model

We consider a simplified model of electric grid, omitting some detailed properties of power systems. Given a network  $G = (N, E)$  representing an electric grid, each node  $s \in N$  is associated with its voltage  $V_s$  and each edge  $(s, s') \in E$  with its current  $I_{(s, s')}$  and impedance  $Z_{(s, s')}$ , satisfying the electricity law  $V_s - V_{s'} = I_{(s, s')}Z_{(s, s')}$ . There is a collection of loads  $\mathcal{N}$ , where each load  $k \in \mathcal{N}$  is attached to a node  $s_k$  in  $V$ . Let  $s_0 \in N$  denote a particular node that represents the ground. For completeness, we assume  $(s_k, s_0) \in E$  for all  $k \in \mathcal{N}$ . To simplify the notations, let  $Z_k \triangleq Z_{(s_k, s_0)}$  be internal impedance between the nodal voltage  $V_{s_k}$ , and the ground. Each  $k \in \mathcal{N}$  requires a power demand  $d_k$  and will return utility  $u_k$ , if satisfied. Similarly, we define  $I_k \triangleq I_{(s_k, s_0)}$  for all  $k \in \mathcal{N}$ . See Fig. 4 for an example.

We require that the current transmitting in each edge is subject to its capacity  $C_{(s, s')}$ . We consider a single source (or generator) at node  $s_G \in N$ . We assume that the generation power is not limited, and hence can feasibly support all loads, if not limited by the edge capacity. The problem can be modelled by the following mathematical program:

$$\text{(NKP)} \quad \max_{x_k \in \{0, 1\}} \sum_{k \in \mathcal{N}} u_k x_k \quad (7)$$

$$\text{subject to} \quad \frac{V_{s_k}}{Z_k} x_k = I_k \text{ for all } k \in \mathcal{N} \quad (8)$$

$$d_k x_k = \frac{V_{s_k}^2}{Z_k} \text{ for all } k \in \mathcal{N} \quad (9)$$

$$V_s - V_{s'} = I_{(s, s')} Z_{(s, s')} \text{ for all } (s, s') \in E \quad (10)$$

$$\sum_{v: (s, v') \in E} I_{(s, v')} = 0 \text{ for all } s \neq s_G \quad (11)$$

$$|I_{(s, s')}| \leq C_{(s, s')} \text{ for all } (s, s') \in E \quad (12)$$

The quantities  $d_k, V_s, I_{(s, s')}, Z_{(s, s')}$  can be complex numbers, while the capacity constraint  $C_{(s, s')}$  is a real number. If all  $V_s, I_{(s, s')}, Z_{(s, s')}$  are real numbers, then this captures a DC (direct-current) system.

#### E. Approximation Algorithms

Given an allocation  $(x_k)_{k \in \mathcal{N}} \in \{0, 1\}^n$ , we equivalently represent it by the satisfied subset of users  $S \subseteq \mathcal{N}$ , where  $S \triangleq \{k \in \mathcal{N} \mid x_k = 1\}$ . For an allocation  $x$ , we denote  $u(x) \triangleq \sum_{k \in \mathcal{N}} u_k x_k$ , also for subset  $S$ , we denote  $u(S) \triangleq \sum_{k \in S} u_k$ . We denote  $S^* \subseteq \mathcal{N}$  to be an optimal solution of CKP and  $\text{OPT} \triangleq \sum_{k \in S^*} u_k$  be the corresponding total utility.

**Definition 3.1:** For  $\alpha \in (0, 1]$  and  $\beta \geq 1$ , a bi-criteria  $(\alpha, \beta)$ -approximation to Eqn. (5) is an allocation  $(\hat{x}_k)_{k \in \mathcal{N}} \in \{0, 1\}^n$  satisfying

$$\left| \sum_{k \in \mathcal{N}} d_k \hat{x}_k \right| \leq \beta \cdot C \quad (13)$$

$$\text{such that} \quad \sum_{k \in \mathcal{N}} u_k \hat{x}_k \geq \alpha \cdot \text{OPT}. \quad (14)$$

For the networked case, an algorithm is called an  $(\alpha, \beta)$ -approximation algorithm to NKP, if it returns a solution

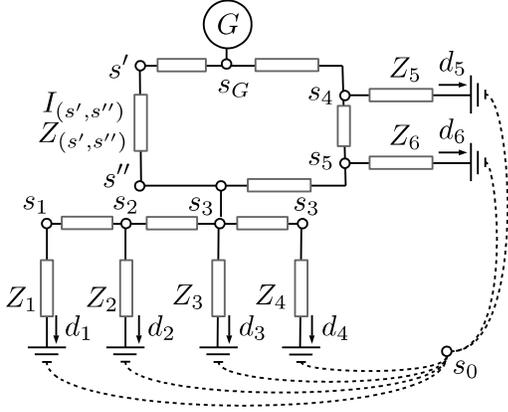


Fig. 4: An example of simplified electric grid.

$\hat{x} \in \{0, 1\}^{\mathcal{N}}$ , such that  $\sum_{k \in \mathcal{N}} u_k \hat{x}_k \geq \alpha \cdot \text{OPT}$  subject to Constraints (8)-(11), and

$$|I_{(u,v)}| \leq \beta \cdot C_{(u,v)} \quad \text{for all } (u,v) \in E$$

In particular, *polynomial-time approximation scheme* (PTAS) is a  $(1 - \epsilon, 1)$ -approximation algorithm for any  $\epsilon > 0^1$ . The running time of a PTAS is polynomial in the input size for every fixed  $\epsilon$ , but the exponent of the polynomial might depend on  $1/\epsilon$ . One way of addressing this is to define the *efficient polynomial-time approximation scheme* (EPTAS), whose running time is the multiplication of a function in  $1/\epsilon$  and a polynomial in the input size independent of  $\epsilon$ .

An even stronger notion is a *fully polynomial-time approximation scheme* (FPTAS), which requires the running time to be polynomial in both input size and  $1/\epsilon$ . In this paper, we are interested in bi-criteria FPTAS, which is a  $(1, 1 + \epsilon)$ -approximation algorithm for any  $\epsilon > 0$ , with the running time to be polynomial in the input size and  $1/\epsilon$ .

#### IV. HARDNESS OF POWER ALLOCATION IN AC ELECTRIC SYSTEMS

In this section, we present our main hardness result for CKP, which depends on the maximum angle  $\phi$  the demands make with the positive real-axis. When  $\phi \in [\frac{\pi}{2} + \epsilon, \pi]$ , we show that the problem is inapproximable within any polynomial factor if we do not allow a violation of Constraint (6). Moreover, when  $\phi$  approaches  $\pi$ , there is no  $(\alpha, \beta)$ -approximation.

*Theorem 4.1:* Unless  $P=NP$ ,

- there is no  $(\alpha, 1)$ -approximation for  $\text{CKP}[\frac{\pi}{2} + \epsilon, \pi]$  where  $\alpha, \epsilon$  have polynomial length.
- there is no  $(\alpha, \beta)$ -approximation for  $\text{CKP}[\pi - \epsilon', \pi]$ , where  $\alpha$  and  $\beta$  have polynomial length, and  $\epsilon'$  depends exponentially on  $n$ .

<sup>1</sup>We note the distinction with  $\epsilon$ , which is associated with an angle.

In fact, these hardness results hold even if we assume that all demands are on the real line, except one demand  $d_{m+1}$  such that  $\arg(d_{m+1}) = \frac{\pi}{2} + \theta$ , for some  $\theta \in [\epsilon, \frac{\pi}{2}]$  (see Fig. 5).

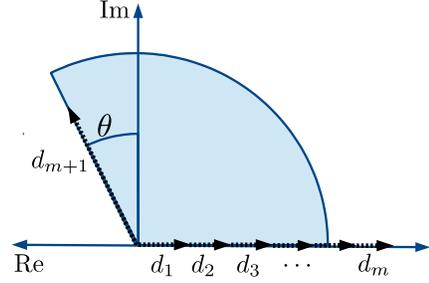


Fig. 5: The left figure show the set of all demands  $\{d_k\}$ . Red vectors represent a feasible solution to (CKP) such that the total magnitude of red demands remain within radius  $C$  as shown in the right figure.

*Proof:* We present a reduction from the (weakly) NP-hard subset sum problem (SUBSUM): given an instance  $I$ , a set of positive integers  $A \triangleq \{a_1, \dots, a_m\}$  and a positive integer  $B$ , does there exist a subset of  $A$  that sums-up to exactly  $B$ ?

Assuming we have an  $(\alpha, \beta)$ -approximation for  $\text{CKP}[\frac{\pi}{2} + \theta, \frac{\pi}{2} + \theta]$ , we construct a  $\text{CKP}[\frac{\pi}{2} + \theta, \frac{\pi}{2} + \theta]$  instance  $I'$  for each instance  $I$  of SUBSUM such that  $\text{SUBSUM}(I)$  is a "yes" instance if and only if the output of the  $(\alpha, \beta)$ -approximation algorithm on the  $\text{CKP}(I')$  instance has utility at least  $\alpha$ . We assume that  $\tan \theta$  is a rational number that can be encoded using a polynomial number of bits.

We define  $n \triangleq m + 1$  demands: for each  $a_k, k = 1, \dots, m$ , define a demand  $d_k \triangleq a_k$ , and an additional demand

$$d_{m+1} \triangleq -B + \mathbf{i}B \cot \theta.$$

For all  $k = 1, \dots, m$ , let utility  $u_k \triangleq \frac{\alpha}{m+1}$ , and  $u_{m+1} \triangleq 1$ . We let

$$C \triangleq B \cot \theta.$$

We prove the first direction, assuming  $\text{SUBSUM}(I)$  is feasible. Namely,  $\sum_{k=1}^m a_k \hat{x}_k = B$ , where  $\hat{x} \in \{0, 1\}^m$  is a solution vector of SUBSUM. Construct a solution  $x \in \{0, 1\}^{m+1}$  of CKP such that

$$x_k = \begin{cases} \hat{x}_k & \text{if } k = 1, \dots, m \\ 1 & \text{if } k = m + 1. \end{cases} \quad (15)$$

In fact, this is a feasible solution that satisfies Constraint (6): using  $\sum_{k=1}^m a_k \hat{x}_k - B = 0$ , we get

$$\left( \sum_{k=1}^m d_k^R x_k + d_{m+1}^R \right)^2 + \left( \sum_{k=1}^m d_k^I x_k + d_{m+1}^I \right)^2 \quad (16)$$

$$= \left( \sum_{k=1}^m d_k^I x_k - B \right)^2 + B^2 \cot^2 \theta \quad (17)$$

$$= B^2 \cot^2 \theta = C^2. \quad (18)$$

Since  $u_{m+1} = 1$ , the total utility of such solution  $u(x) \geq 1$ , which implies that OPT is at least 1, and hence by the feasibility of this solution, any  $(\alpha, \beta)$ -approximation algorithm would return a solution of utility at least  $\alpha$ .

Conversely, assume that the  $(\alpha, \beta)$ -approximation algorithm gives a solution  $x \in \{0, 1\}^{m+1}$  of utility at least  $\alpha$ . Since user  $m+1$  has utility  $u_{m+1} = 1$ , while the rest of users utilities total to less than  $\alpha$ :  $\sum_{k=1}^m u_k < \alpha$ , user  $m+1$  must be included in this solution. Therefore, substituting in Constraint (13),

$$\left( \sum_{k=1}^m d_k^R x_k - B \right)^2 + B^2 \cot^2 \theta \leq \beta^2 C^2 \quad (19)$$

gives

$$\begin{aligned} \left( \sum_{k=1}^m a_k x_k - B \right)^2 &\leq \beta^2 C^2 - B^2 \cot^2 \theta \\ &= B^2 \cot^2 \theta (\beta^2 - 1). \end{aligned} \quad (20)$$

By the integrality of the  $a_i$ 's,

$$\sum_{k=1}^m a_k x_k = B \iff \left| \sum_{k=1}^m a_k x_k - B \right| < 1 \quad (21)$$

In other words, SUBSUM is feasible if and only if the absolute difference  $\left| \sum_{k=1}^m a_k x_k - B \right| < 1$ . This implies, SUBSUM( $I$ ) is feasible when the R.H.S. of Eqn. (20) is strictly less than 1. When  $\beta = 1$ , R.H.S. of Eqn. (20) is zero; and we complete the second direction and hence, the first part of the proof.

Eqn. (21) also holds when  $\theta$  is large enough such that the R.H.S. of Eqn. (20)

$$B^2 \cot^2 \theta (\beta^2 - 1) < 1,$$

This implies,  $\theta > \tan^{-1} \sqrt{B^2(\beta^2 - 1)}$ , where  $B$  is not polynomial in  $n$ . This completes the second direction and the second part of the proof. ■

### Discussion

For completeness, we describe briefly the approximation algorithms presented in [2]. We start with the  $(1 - \epsilon)$ -approximation PTAS algorithm for CKP $[0, \frac{\pi}{2}]$ . The algorithm enumerates through all subsets of demands of size at most  $\frac{1}{\epsilon}$ . In each step, we define a polygonized region as shown in Fig. 6-(a) as the thick dark lines. Next, we define a perpendicular line from the origin to each segment of the polygonized region. By projecting all demands  $d_k$  on each of the segments, we define an  $m$ -dimensional knapsack (MDKP) problem [3] such that each demand now is replaced by its projected values, and  $m$  is the number of segments. Using the PTAS for MDKP [3], one can eventually obtain  $(1 - \epsilon)$ -approximation for CKP $[0, \frac{\pi}{2}]$  by repeating this procedure for all possible set of demands of size at most  $\frac{1}{\epsilon}$ . The main challenge is how to define the granularity of the lattice at each enumeration. [2] proved that a granularity of  $\frac{\epsilon W^R}{4}$  and  $\frac{\epsilon W^I}{4}$  on the real and imaginary axis respectively is sufficient (see Fig. 6-(a)).

Next, we describe the  $(1, 1 + \epsilon)$ -approximation bi-criteria FPTAS algorithm for CKP $[\frac{\pi}{2}, \pi - \epsilon]$ . The basic idea is to

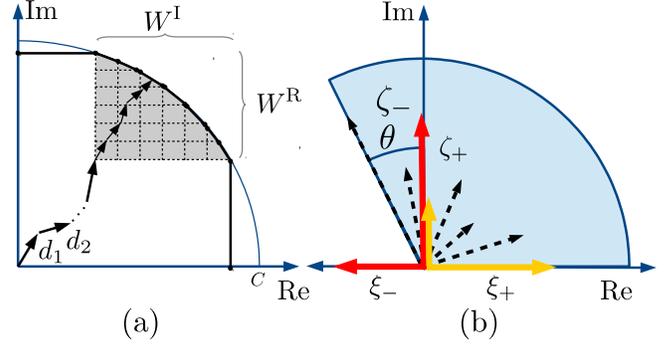


Fig. 6: (a) an illustration for the polygonizing procedure for CKP $[0, \frac{\pi}{2}]$ . (b) CKP $[0, \pi - \epsilon]$ : for each quarter, we define a separate 2DKP problem. In the first quarter, the total real part of demands should be exactly  $\xi_+$ , while the imaginary part should be  $\zeta_+$ . The same procedure applies for demands in the second quarter, but using respectively  $\xi_-$  and  $\zeta_-$ .

round all demands by  $L \triangleq \frac{\epsilon}{n(\tan \theta + 1)}$  and then partition them according to the quarter they reside in. For each of the two quarters, we enumerate over all possible horizontal and vertical projections of the new demands such that the sum of projections are at most  $C$  in each of the real and imaginary axes. Then we solve two separate 2DKP problem using dynamic programming. Fig. 6-(b) presents a pictorial illustration.

## V. HARDNESS OF POWER ALLOCATION IN ELECTRIC NETWORKS

In this section, we present a hardness result showing that there exists no bi-criteria approximation algorithms for NKP, even all power demands are real numbers. We note that the hardness result holds even for only DC demands, when one demand is negative and the rest are positive.

*Theorem 5.1:* Unless P=NP, there is no  $(\alpha, \beta)$ -approximation for NKP where  $\alpha$  and  $\beta$  have polynomial length, even all power demands are real numbers (i.e.,  $\text{Im}(d_k) = 0$  for all  $k \in \mathcal{N}$ ).

*Proof:* We present a reduction from the (weakly) NP-hard subset sum problem (SUBSUM): given a set of positive integers  $A \triangleq \{a_1, \dots, a_m\}$  and a positive integer  $B$ , does there exist a subset of  $A$  that sums-up to exactly  $B$ ?

Assuming we have an  $(\alpha, \beta)$ -approximation algorithm for NKP, we construct a NKP instance  $I'$  for each instance  $I$  of SUBSUM such that SUBSUM( $I$ ) is a "yes" instance if and only if the utility of NKP( $I'$ ) is at least  $\alpha$ .

We consider the gadget in Fig. 7, where we let

$$\begin{aligned} C_{(a,b)} &= C, & C_{(G,a)} &= C_{(G,b)} = \infty, & \text{and} \\ Z_{(G,a)} &= Z_{(G,b)} &= Z_{(a,b)} &= 1. \end{aligned}$$

For each  $a_k, k = 1, \dots, m$ , define a load with  $Z_k = \frac{1}{m+1}$  and utility  $u_k = \frac{\alpha}{m+1}$ . Then, we create an additional load  $Z_{m+1} =$

$\frac{1}{B}$  and utility  $u_{m+1} = 1$ . Finally, let  $V_G = 1$  and

$$C \triangleq \frac{1}{\beta(3 + 2B + (2 + B)\sum_{k=1}^n a_k + \epsilon)}, \quad \text{for some } \epsilon > 0. \quad (22)$$

Let  $x = (x_1, \dots, x_{m+1})$  be the solution returned by the  $(\alpha, \beta)$ -approximation algorithm. If the total utility of  $x$  is at least  $\alpha$ , then we necessarily have  $x_{m+1} = 1$ . By the balance law of electricity flows, we obtain

$$\begin{aligned} V_G - V_a &= V_a - V_b + \sum_{k=1}^m \frac{V_a x_k}{Z_k} \quad \text{and} \\ V_G - V_b + V_a - V_b &= \frac{V_b x_{m+1}}{Z_{m+1}}. \end{aligned} \quad (23)$$

Hence, we have

$$\frac{V_a}{V_b} = \frac{3 + \frac{x_{m+1}}{Z_{m+1}}}{3 + \sum_{k=1}^m \frac{x_k}{Z_k}}. \quad (24)$$

Substituting in Eqn. (23) and solving for  $V_b$ , we obtain

$$V_b = \frac{V_G(3 + \sum_{k=1}^m \frac{x_k}{Z_k})}{3 + 2\frac{x_{m+1}}{Z_{m+1}} + \left(2 + \frac{x_{m+1}}{Z_{m+1}}\right) \sum_{k=1}^m \frac{x_k}{Z_k}}. \quad (25)$$

Since  $Z_{(a,b)} = 1$ , the relaxed capacity constraint implies  $|V_a - V_b| \leq \beta C$ , which, by Eqn. (24) and Eqn. (25), is equivalent to

$$\begin{aligned} \left| \frac{x_{m+1}}{Z_{m+1}} - \sum_{k=1}^m \frac{x_k}{Z_k} \right| &\leq \beta C \left( 3 + \frac{2x_{m+1}}{Z_{m+1}} + \left(2 + \frac{x_{m+1}}{Z_{m+1}}\right) \sum_{k=1}^m \frac{x_k}{Z_k} \right) \\ \Rightarrow \left| B - \sum_{k=1}^m a_k x_k \right| &< 1, \end{aligned} \quad (26)$$

where the implication follows from the choice of  $C$ . Thus we conclude that the SUBSUM instance  $I$  is feasible.

Conversely, given a feasible solution  $x \in \{0, 1\}^{m+1}$  satisfying  $\sum_{k=1}^n a_k x_k - B x_{m+1} = 0$ , with  $x_{m+1} = 1$ , we can see from Eqn. (26) that  $x$  is a feasible solution (with  $\beta = 1$ ) to the NKP instance  $I'$ , which has utility at least 1. Thus the  $(\alpha, \beta)$ -approximation algorithm returns a solution of utility at least  $\alpha$ . ■

## VI. CONCLUSION AND FUTURE WORK

In this paper we showed that the algorithms provided in [2] are the best approximation possible for CKP problem. As a future work, we can focus on a general setting of knapsack problem for AC electric systems with a combination of capacity constraints on apparent power, active power and reactive power together. We call this problem general complex-demand knapsack problem (GCKP).

$$\text{(GCKP)} \quad \max_{x_k \in \{0,1\}} \sum_{k \in \mathcal{N}} u_k x_k \quad (27)$$

$$\text{subject to} \quad \left| \sum_{k \in \mathcal{N}} x_k d_k \right| \leq C', \quad (28)$$

$$\sum_{k \in \mathcal{N}} x_k d_k^R \leq C^R, \quad (29)$$

$$\sum_{k \in \mathcal{N}} x_k d_k^I \leq C^I. \quad (30)$$

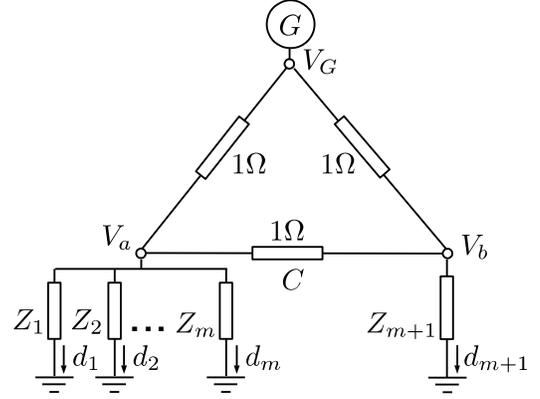


Fig. 7: A gadget for the reduction from SUBSUM to NKP.

Another future direction is to extend CKP study to a restricted electric grid setting. In the simplified model of electric grid, the inapproximability of Theorem 5.1 implies that the possibility of approximation algorithms in a networked setting will require a further restriction of the network topology, for instance, considering networks without cycles.

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